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FREEZING A SATURATED LIQUID INSIDE A SPHERE

JAMES M. HILL and ADAM KUCERA
Department of Mathematics, University of Wollongong, P.O. Box 1144,
Wollongong, NSW 2500, Australia

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Abstract—A new semi-analytical procedure is employed for the problem of freezing a saturated liquid inside a spherical container and including the effect of radiation at the container surface. Generally moving boundary problems of this type involve a boundary layer analysis. The approximation scheme employed here avoids this complication and gives rise to successive estimates of the time t_c for complete solidification of the sphere. In addition an integral formulation is adopted to independently establish bounds for t_c. The upper and lower bounds obtained are the standard order one corrected estimate of t_c and the pseudo steady-state estimate of t_c, respectively. Numerical values of the successive estimates for t_c indicate firstly satisfactory convergence and secondly that they are indeed consistent with the bounds independently established. Numerical values obtained for the position of the moving front are in agreement with previous results arising from both a completely numerical solution and an alternative semi-analytical solution of the problem.

NOMENCLATURE

a	radius of spherical container
$A_n(x)$	functions appearing in assumed expansion for $\phi(x, y)$
C	heat capacity of solid
h	heat transfer coefficient
k	thermal conductivity of solid
L	latent heat of fusion
$p_n(x)$	polynomial expression in x of degree $n-2$ $(n \geqslant 2)$
r^*	position radius
r	dimensionless position radius
$R^*(t^*)$	radius of moving front
$R(t)$	dimensionless radius of moving front
t^*	time
t	dimensionless time
t_c	time to complete solidification
t_{1c}	pseudo steady-state estimate of t_c
t_{2c}	first-order corrected estimate of t_c
$\hat{t}_{1c}, \hat{t}_{2c}, \hat{t}_{3c}$	approximating estimates to t_c deduced from solution
T_f	fusion temperature
T_0	coolant temperature
$T^*(r^*, t^*)$	temperature of solid
$T(r, t)$	dimensionless temperature
$T_1(r, t)$	pseudo steady-state estimate for $T(r, t)$
$T_2(r, t), T_3(r, t)$	higher order corrections to pseudo steady-state estimate
$u(r, t)$	new dependent variable defined by equation (8)
x, y	new independent variables defined by equation (7)

Greek symbols

α, β	constants defined by equation (4)
γ, δ	constants defined by equation (16)
λ	variable defined by equation (22)
ξ	dummy integration variable
ρ	density of solid
$\phi(x, y)$	new dependent variable defined by equation (9)
$\phi_1(x, y)$	pseudo steady-state estimate for $\phi(x, y)$
$\phi_2(x, y)$	first-order correction to pseudo state-state estimate
ω	variable defined by equation (31)

INTRODUCTION

THE MOVING boundary problem associated with freezing liquids inside containers is relevant in many industrial processes such as casting thermoplastics or metals, freezing foods and producing ice. Generally such problems do not admit closed analytical solutions and the governing equations must be solved either numerically or by an approximate semi-analytical procedure. The problems of freezing a saturated liquid inside a sphere and including radiation at the surface has been studied by a number of authors [1–3]. Semi-analytical techniques are employed in refs. [1, 2] while a fully numerical treatment is given in ref. [3]. In general owing to the occurrence of a thermal boundary layer as the moving front approaches the centre of the sphere, the full mathematical analysis of such problems involves a fairly complicated asymptotic boundary layer approach (see for example refs. [4–6]). Here we present a new approximate analytical solution to the problem which is meaningful up to and including the time t_c to complete solidification and therefore avoids a boundary layer analysis. Moreover, we independently establish upper and lower bounds for t_c.

Consider a molten material inside a spherical container of radius a and at its uniform fusion temperature T_f. Suppose the container is surrounded by a coolant which is maintained at constant temperature T_0 then assuming constant physical properties of the solid and negligible volume change in solidification the temperature $T^*(r^*, t^*)$ of the solid and the radius $R^*(t^*)$ of the moving front can be described by the following system:

$$\frac{\partial T^*}{\partial t^*} = \frac{k}{\rho C}\left(\frac{\partial^2 T^*}{\partial r^{*2}} + \frac{2}{r^*}\frac{\partial T^*}{\partial r^*}\right), \quad R^*(t^*) < r^* < a,$$

$$-k\frac{\partial T^*}{\partial r^*}(a,t^*) = h\{T^*(a,t^*) - T_0\}, \qquad (1)$$

$$T^*[R^*(t^*),t^*] = T_f, \quad \frac{\partial T^*}{\partial r^*}[R^*(t^*),t^*] = \frac{\rho L}{k}\frac{dR^*}{dt^*},$$

where $R^*(0) = a$. In terms of the following dimensionless variables

$$r = \frac{r^*}{a}, \quad R = \frac{R^*}{a}, \quad t = \frac{kt^*}{\rho C a^2}, \qquad (2)$$

$$T = \frac{T_f - T^*}{T_f - T_0},$$

system (1) for $T(r,t)$ and $R(t)$ becomes

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}, \quad R(t) < r < 1,$$

$$T(1,t) + \beta\frac{\partial T}{\partial r}(1,t) = 1, \qquad (3)$$

$$T[R(t),t] = 0, \quad \frac{\partial T}{\partial r}[R(t),t] = -\alpha\frac{dR}{dt},$$

where $R(0) = 1$ and the constants α and β are defined by

$$\alpha = \frac{L}{C(T_f - T_0)}, \quad \beta = \frac{k}{ha}. \qquad (4)$$

Clearly since $T_0 < T^* < T_f$ we have $0 < T < 1$ for $R(t) < r < 1$.

In the following section we outline a procedure for obtaining an approximate analytical solution of system (3). This procedure is based on one given recently by Davis and Hill [7] for system (3) with the constant β identically zero (that is, no radiation at the surface). The case $\beta = 0$ was originally studied using a regular perturbation series of the form [8]

$$T(r,t) = T_1(r,t) + T_2(r,t)/\alpha + T_3(r,t)/\alpha^2 + O(\alpha^{-3}). \quad (5)$$

However, the terms of order α^{-1} and α^{-2} are singular as $R(t)$ approaches zero and consequently a full boundary layer analysis is required (see refs. [4–6, 9]). By appropriate choice of variables this difficulty is avoided in ref. [7] and it would seem worthwhile applying the method of ref. [7] to the case when the constant β is strictly non-zero. We find that the mathematical details are quite different to those given in ref. [7] and moreover, the results of ref. [7] are not contained as a special case in the present analysis.

In subsequent sections we summarize the main results of the calculation leading to the motion of the moving boundary and we indicate how the result obtained relates to the standard pseudo steady-state and order one corrected motions. We find that these latter approximations to the motion arise from the analysis given here by assuming that certain terms in the solution are in fact the first few terms of a geometric series which on summation yields the required results. For completeness we also give an independent derivation of the pseudo steady-state and order one corrected motions which are formally obtained by assuming a series expansion of the form of equation (5) and consistently neglecting terms of order α^{-2} and higher in comparison to the two leading terms.

From an independent integral formulation of the problem similar to those given by Shih and Chou [2] and Theofanous and Lim [10] we prove that t_c satisfies the inequalities

$$(1+2\beta)\alpha/6 \leqslant t_c \leqslant (1+2\beta)(\alpha+1)/6. \qquad (6)$$

We observe that the lower bound is merely the pseudo steady-state estimate for the time to complete solidification while the upper bound is the order one corrected estimate formally obtained from the first two terms of series (5). The relevant details of this proof are noted in a subsequent section. In the final section numerical values for the position $R(t)$ of the moving front are shown to be in close agreement with those of Shih and Chou [2] and Tao [3].

METHOD OF SOLUTION

In this section we describe briefly the solution procedure which involves reducing the moving boundary problem (3) to one with fixed boundaries. We introduce new independent variables x and y defined by

$$x = (r-1)/[R(t)-1], \quad y = R(t), \qquad (7)$$

and after making the standard transformation

$$T(r,t) = u(r,t)/r, \qquad (8)$$

we suppose

$$u(r,t) = \phi(x,y). \qquad (9)$$

It is now a simpler matter to show that system (3) becomes

$$\alpha y\frac{\partial^2\phi}{\partial x^2} = \frac{\partial\phi}{\partial x}(1,y)\left[x\frac{\partial\phi}{\partial x} - (y-1)\frac{\partial\phi}{\partial y}\right],$$

$$\beta\frac{\partial\phi}{\partial x}(0,y) + (1-\beta)(y-1)\phi(0,y) = (y-1), \quad (10)$$

$$\phi(1,y) = 0, \quad \frac{\partial\phi}{\partial x}(1,y) = -\alpha y(y-1)\frac{dy}{dt},$$

and $y(0) = 1$. In $(10)_1$ the arguments of ϕ and its partial derivatives are understood to be (x,y) unless otherwise indicated. Moreover, we remark that in the derivation of $(10)_1$ we have utilized $(10)_4$.

The above non-linear system can be formally solved by assuming a series solution for $\phi(x,y)$ of the form

$$\phi(x,y) = \sum_{n=0}^{\infty} A_n(x)(y-1)^n, \qquad (11)$$

where $A_n(x)$ denote functions of x only. From equations (10) and (11) we find that the functions $A_n(x)$ are determined by solving

$$A_0'' = \alpha^{-1}A_0'(1)xA_0',$$

$$A_n''+A_{n-1}''=\frac{1}{\alpha}\sum_{j=0}^{n} A_j'(1)[xA_{n-j}'-(n-j)A_{n-j}] \quad (n \geq 1), \tag{12}$$

subject to the boundary conditions

$$\begin{gathered} A_0'(0) = 0, \quad A_n(1) = 0 \quad (n \geq 0), \\ \beta A_1'(0)+(1-\beta)A_0(0) = 1, \\ \beta A_n'(0)+(1-\beta)A_{n-1}(0) = 0 \quad (n \geq 2), \end{gathered} \tag{13}$$

where primes denote differentiation with respect to x and in (12) the argument of A_n is understood to be x unless otherwise indicated. From $(10)_4$ and (11) we find that the motion of the moving front is obtained from

$$\sum_{n=0}^{\infty} A_n'(1)(y-1)^n = -\alpha y(y-1)\frac{dy}{dt}, \tag{14}$$

and $y(0) = 1$. From (12) and (13) we find that $A_0(x)$ is identically zero while the remaining $A_n(x)$ are obtained simply by integrating equations of the form

$$A_n''(x) = p_n(x), \tag{15}$$

where $p_n(x)$ denotes a polynomial expression in x of degree $n-2$ [$n \geq 2, p_1(x) = 0$]. This is in contrast to the problem $\beta = 0$ which gives rise to confluent hypergeometric solutions for $A_n(x)$ (see ref. [7]). The final results for $A_n(x)$ are given in the following section.

SUMMARY OF RESULTS

Introducing new constants γ and δ defined by

$$\gamma = \alpha(\beta-1), \quad \delta = 2-\alpha, \tag{16}$$

we find that the first six functions $A_n(x)$ are as follows:

$$A_0(x) = 0, \quad A_1(x) = \frac{(x-1)}{\beta}, \quad A_2(x) = \frac{(x-1)}{2\alpha\beta^2}(x+1-2\gamma),$$

$$A_3(x) = \frac{(x-1)}{6\alpha^2\beta^3}\{\gamma x^2+(3\delta-11\gamma)x+(3\delta-14\gamma+6\gamma^2)\},$$

$$\begin{aligned} A_4(x) = \frac{(x-1)}{24\alpha^3\beta^4}\{&(4\gamma-\delta)x^3+(4\gamma-\delta)(1-4\gamma)x^2 \\ &+(104\gamma^2-56\gamma\delta-68\gamma+5\delta+12\delta^2+12)x-(24\gamma^3-160\gamma^2+68\gamma\delta+68\gamma-5\delta-12\delta^2-12)\}, \end{aligned}$$

$$\begin{aligned} A_5(x) = \frac{(x-1)}{120\alpha^4\beta^5}\{&\gamma(4\gamma-\delta)x^4+(40\gamma+79\gamma\delta-136\gamma^2-10-15\delta^2)x^3 \\ &+(60\gamma+200\gamma^3+89\gamma\delta-256\gamma^2-10-15\delta^2+20\gamma\delta^2-100\gamma^2\delta)x^2 \\ &+(-1000\gamma^3-940\gamma-891\gamma\delta+2264\gamma^2+800\gamma^2\delta-340\gamma\delta^2+60\delta^3+75\delta^2+220\delta+50)x \\ &+(120\gamma^4-1800\gamma^3+1140\gamma^2\delta-1000\gamma-916\gamma\delta-400\gamma\delta^2+2604\gamma^2+60\delta^3+75\delta^2+220\delta+50)\}. \end{aligned} \tag{17}$$

In particular the values of $A_n'(1)$ which are needed to determine the motion of the moving front are given by

$$\begin{aligned} &A_0'(1) = 0, \quad A_1'(1) = \beta^{-1}, \\ &A_2'(1) = (1-\gamma)/\alpha\beta^2, \quad A_3'(1) = (\gamma^2-4\gamma+\delta)/\alpha^2\beta^3, \\ &A_4'(1) = (\delta+3\delta^2+3-15\gamma\delta-3\gamma^3+31\gamma^2-16\gamma)/3\alpha^3\beta^4, \\ &A_5'(1) = (3\gamma^4-65\gamma^3+112\gamma^2-46\gamma+3\delta^3 \\ &\qquad +3\delta^2+2+11\delta-41\gamma\delta+46\gamma^2\delta-18\gamma\delta^2)/3\alpha^4\beta^5. \end{aligned} \tag{18}$$

We make use of these results in the following section to determine an approximate expression for the motion of the moving front.

MOTION OF THE MOVING BOUNDARY

From (14), (18) and the initial condition $y(0) = 1$ we can deduce the following series for the motion of the moving boundary

$$\begin{aligned} t = -\alpha\beta\Big\{&(y-1)+(1+2\gamma-\delta)\frac{(y-1)^2}{2\alpha\beta} \\ &+(\gamma^2+3\gamma-1-\gamma\delta)\frac{(y-1)^3}{3\alpha^2\beta^2}+(4\gamma-\gamma^2-\delta)\frac{(y-1)^4}{3\alpha^3\beta^3} \\ &+(5\gamma^3-52\gamma^2+28\gamma-5-5\delta^2-2\delta+25\gamma\delta) \\ &\times\frac{(y-1)^5}{15\alpha^4\beta^4}+\ldots\Big\}. \end{aligned} \tag{19}$$

In particular an approximating expression for the time t_c to complete solidification can be obtained from (19) by setting y zero, thus

$$\begin{aligned} t_c = \alpha\beta\Big\{&1-\frac{(1+2\gamma-\delta)}{2\alpha\beta} \\ &+\frac{(\gamma^2+3\gamma-1-\gamma\delta)}{3\alpha^2\beta^2}-\frac{(4\gamma-\gamma^2-\delta)}{3\alpha^3\beta^3} \\ &+\frac{(5\gamma^3-52\gamma^2+28\gamma-5-5\delta^2-2\delta+25\gamma\delta)}{15\alpha^4\beta^4}+\ldots\Big\}. \end{aligned} \tag{20}$$

Numerical values of various estimates for t_c obtained from (20) are given in the final section.

We observe from (19) on using (16) and neglecting terms of order α^{-1} and higher

$$t = -\alpha\beta\left\{(y-1)+\frac{(2\beta-1)}{2\beta}(y-1)^2+\frac{(\beta-1)}{3\beta}(y-1)^3\right\} + \left\{\frac{(y-1)^2}{2}-\frac{(\beta-1)(y-1)^3}{3\beta}[1-\lambda+\lambda^2+\ldots]\right\}+o(1), \tag{21}$$

where λ is defined by

$$\lambda = (\beta-1)(y-1)/\beta. \tag{22}$$

It is of interest to note that the exact motion correct up to order α^{-1} can be deduced from (21) by assuming that the series in the square brackets involving λ is the geometric series with sum $(1+\lambda)^{-1}$ (assuming $|\lambda| < 1$). Simplifying the result we obtain from (21)

$$t = -\frac{\alpha}{6}\{2(\beta-1)y^3+3y^2-(2\beta+1)\} + \frac{(y-1)^2}{6}\left\{\frac{(2\beta+1)+(\beta-1)y}{1+(\beta-1)y}\right\}+o(1), \tag{23}$$

which can of course be deduced more directly from an assumption of the form (5). For completeness the essential details of such a calculation are given in the following section using the (x, y) variables and equation (10) for $\phi(x, y)$.

Finally in this section we observe that the first term of (23) is precisely the pseudo steady-state result and moreover that the pseudo steady-state expression for $\phi(x, y)$ can be seen to arise from (11) and (17) by retaining only the terms of order one in expressions (17). We find that

$$\phi(x,y) = \frac{(x-1)(y-1)}{\beta}\{1-\lambda+\lambda^2-\lambda^3+\lambda^4+\ldots\}+o(1), \tag{24}$$

where λ is given by (22). Again on assuming the series involving λ is the geometric series and that $|\lambda| < 1$ we obtain in a straightforward manner

$$\phi(x,y) = \frac{(x-1)(y-1)}{[1+(\beta-1)y]}+o(1), \tag{25}$$

which is precisely the pseudo steady-state result.

DIRECT DERIVATION OF FIRST ORDER CORRECTION

If for large α we assume

$$\phi(x,y) = \phi_1(x,y)+\phi_2(x,y)/\alpha+o(\alpha^{-1}), \tag{26}$$

where $\phi_1(x,y)$ is the pseudo steady-state result given in (25) then from (10) we find that $\phi_2(x,y)$ is obtained by solving

$$\frac{\partial^2\phi_2}{\partial x^2} = \frac{(y-1)^2[\beta+(\beta-1)x(y-1)]}{y[1+(\beta-1)y]^3}, \qquad \beta\frac{\partial\phi_2}{\partial x}(0,y)+(1-\beta)(y-1)\phi_2(0,y) = 0, \quad \phi_2(1,y) = 0. \tag{27}$$

The final result for $\phi_2(x,y)$ is

$$\phi_2(x,y) = \frac{(x-1)(y-1)^2}{6y[1+(\beta-1)y]^4} \times \{(\beta-1)x^2(y-1)[1+(\beta-1)y] + [1+(\beta-1)y][(2\beta+1)+(\beta-1)y]x + \beta[(2\beta+1)+(\beta-1)y]\}. \tag{28}$$

Now from $(10)_4$ and (26) we have

$$\frac{\partial\phi_1}{\partial x}(1,y)+\frac{1}{\alpha}\frac{\partial\phi_2}{\partial x}(1,y) = -\alpha y(y-1)\frac{dy}{dt}, \tag{29}$$

which using the above results simplifies to give

$$\frac{1}{\omega}+\frac{(y-1)(\beta^2+\beta\omega+\omega^2)}{3\alpha y\omega^4} = -\alpha y\frac{dy}{dt}, \tag{30}$$

where ω is defined by

$$\omega = 1+(\beta-1)y. \tag{31}$$

On rearranging equation (30), neglecting terms of order α^{-2} and higher and using

$$\beta^2+\beta\omega+\omega^2 = \left(\frac{\beta^3-\omega^3}{\beta-\omega}\right), \tag{32}$$

we can deduce

$$dt = \left\{-\alpha y\omega+\frac{1}{3(\beta-1)}\left[\omega-\frac{\beta^3}{\omega^2}\right]\right\}dy+o(1). \tag{33}$$

This equation readily integrates to yield precisely equation (23). The estimates for the time to complete solidification arising from the pseudo steady-state assumption and the first order correction are discussed further in the final section.

UPPER AND LOWER BOUNDS FOR t_c

On multiplying $(3)_1$ by r^2, integrating the resulting equation with respect to r from $R(t)$ to r and using $(3)_4$ we obtain after a division by r^2

$$\frac{\partial T}{\partial r}(r,t)+\alpha\left(\frac{R(t)}{r}\right)^2\frac{dR}{dt} = \int_{R(t)}^{r}\left(\frac{\xi}{r}\right)^2\frac{\partial T}{\partial t}(\xi,t)\,d\xi. \tag{34}$$

A further integration of this equation with respect to r from $R(t)$ to r and using $(3)_3$ yields

$$T(r,t) = \left\{\alpha R(t)^2\left(\frac{1}{r}-\frac{1}{R(t)}\right) + \frac{\partial}{\partial R}\left(\int_{R(t)}^{r}\xi^2\left(\frac{1}{\xi}-\frac{1}{r}\right)T(\xi,t)\,d\xi\right)\right\}\frac{dR}{dt}, \tag{35}$$

where $\partial/\partial R$ denotes partial differentiation with respect to R with r and R as independent variables. From (34), (35) and the surface boundary condition $(3)_2$ we can deduce

$$\left\{\alpha R(t)[(1-\beta)R(t)-1] + \frac{\partial}{\partial R}\left(\int_{R(t)}^{1} \xi[(\beta-1)\xi+1]T(\xi,t)\,\mathrm{d}\xi\right)\right\}\frac{\mathrm{d}R}{\mathrm{d}t} = 1. \quad (36)$$

From equations (35) and (36) on eliminating $\mathrm{d}R/\mathrm{d}t$ we obtain the following integral formulation of the problem, namely

$$T(r,t) = \frac{\left\{\alpha R(t)^2\left(\frac{1}{r}-\frac{1}{R(t)}\right)+\frac{\partial}{\partial R}\left(\int_{R(t)}^{r}\xi^2\left(\frac{1}{\xi}-\frac{1}{r}\right)T(\xi,t)\,\mathrm{d}\xi\right)\right\}}{\left\{\alpha R(t)[(1-\beta)R(t)-1]+\frac{\partial}{\partial R}\left(\int_{R(t)}^{1}\xi[(\beta-1)\xi+1]T(\xi,t)\,\mathrm{d}\xi\right)\right\}}, \quad (37)$$

which is similar, although not identical to integral formulations given in ref. [2] for $\beta \neq 0$ and in ref. [10] for $\beta = 0$. We also observe that the pseudo steady-state solution for the problem emerges immediately from equation (37) by retaining only the α terms in both the numerator and the denominator of equation (37).

More importantly we observe that equation (36) can be formally integrated immediately to give the following expression for the motion of the moving boundary, namely

$$t = \frac{\alpha}{6}[2(1-\beta)R(t)^3-3R(t)^2+(1+2\beta)] + \int_{R(t)}^{1}\xi[(\beta-1)\xi+1]T(\xi,t)\,\mathrm{d}\xi, \quad (38)$$

where we have used $R(0) = 1$. Thus in particular for t_c we have

$$t_c = \frac{\alpha}{6}(1+2\beta)+\int_0^1 \xi[(\beta-1)\xi+1]T(\xi,t_c)\,\mathrm{d}\xi, \quad (39)$$

and the inequalities (6) follow immediately from equation (39) on making use of $0 \leqslant T(r,t) \leqslant 1$ and noting that β is non-negative.

Table 1. Numerical values of estimates defined by equations (40) and (41) for various values of α and $\beta = 5.0$

α	$\hat{t}_{1c}$ (40)	$\hat{t}_{2c}$ (40)	$\hat{t}_{3c}$ (40)	t_{1c} (41)	t_{2c} (41)
0.5	1.55	1.42	1.17	0.92	2.75
1.0	2.53	2.55	2.38	1.83	3.67
2.0	4.40	4.51	4.47	3.67	5.50
5.0	9.92	10.09	10.17	9.17	11.00
10.0	19.09	19.28	19.41	18.33	20.17
100.0	184.1	184.3	184.5	183.3	185.2

7. NUMERICAL RESULTS

From (20) we obtain the following estimates for the time t_c to complete solidification

$$\begin{aligned}
\hat{t}_{1c} &= \alpha\beta\left\{1-\frac{(1+2\gamma-\delta)}{2\alpha\beta}+\frac{(\gamma^2+3\gamma-1-\gamma\delta)}{3\alpha^2\beta^2}\right\},\\
\hat{t}_{2c} &= \alpha\beta\left\{1-\frac{(1+2\gamma-\delta)}{2\alpha\beta}+\frac{(\gamma^2+3\gamma-1-\gamma\delta)}{3\alpha^2\beta^2}-\frac{(4\gamma-\gamma^2-\delta)}{3\alpha^3\beta^3}\right\},\\
\hat{t}_{3c} &= \alpha\beta\left\{1-\frac{(1+2\gamma-\delta)}{2\alpha\beta}+\frac{(\gamma^2+3\gamma-1-\gamma\delta)}{3\alpha^2\beta^2}-\frac{(4\gamma-\gamma^2-\delta)}{3\alpha^3\beta^3}+\frac{(5\gamma^3-52\gamma^2+28\gamma-5-5\delta^2-2\delta+25\gamma\delta)}{15\alpha^4\beta^4}\right\}.
\end{aligned} \quad (40)$$

From (23) with $y = 0$ we obtain the pseudo steady-state and first order corrected estimates to t_c, namely

$$t_{1c} = \alpha(2\beta+1)/6, \quad t_{2c} = (\alpha+1)(2\beta+1)/6. \quad (41)$$

Numerical values of the above estimates are given in Table 1 for various values of α and $\beta = 5.0$. Firstly these

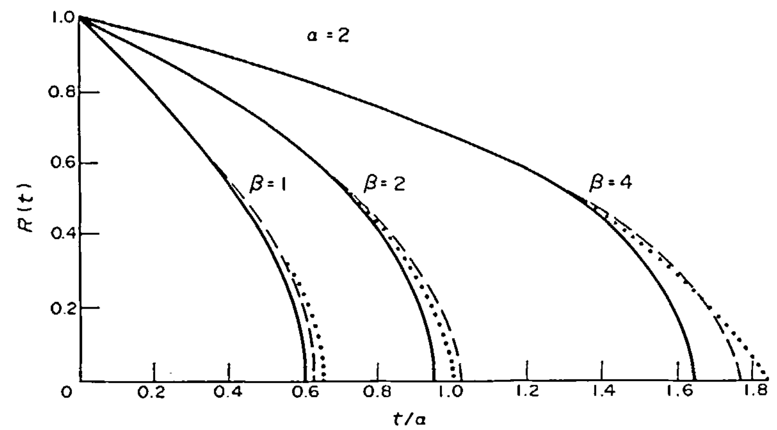

FIG. 1. Variation in $R(t)$ as given by (19) with t/α (for $\alpha = 2$ and three values of β). This work ········, Shih and Chou [2] ———, Tao [3] — — —.

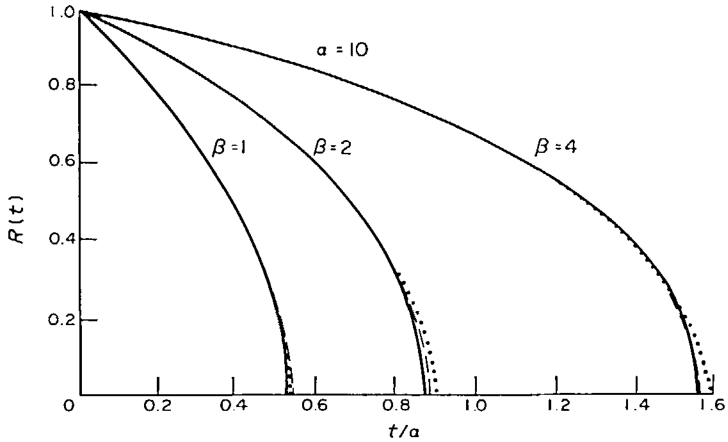

FIG. 2. Variation in $R(t)$ as given by (19) with t/α (for $\alpha = 10$ and three values of β). This work ········, Shih and Chou [2] ———, Tao [3] — — —.

results indicate reasonable convergence of the estimates $\hat{t}_{1c}$, $\hat{t}_{2c}$ and $\hat{t}_{3c}$ and secondly they are consistent with t_{2c} and t_{1c} as upper and lower bounds, respectively. As far as the authors are aware these bounds have not been noted previously and may be useful in a practical context.

Figures 1 and 2 show the variation of position $R(t)$ with time for $\alpha = 2$ and 10, respectively, and three values of β. The numerical values based on (19) are clearly in general agreement with those of refs. [2, 3]. We note, however, the discrepancies for the time to complete solidification with this work and Shih and Chou [2] in the case $\alpha = 2$ and $\beta = 4$. This observation is consistent with comments of Shih and Chou [2] who indicate their semi-analytical procedure is more accurate for small β. In contrast series (19) and (20) are seen to converge more rapidly for large β and therefore in this sense the procedure described here is complementary to that of Shih and Chou [2].

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CONGELATION D'UN LIQUIDE SATURE DANS UNE SPHERE

Résumé—On emploie une procédure semi-analytique pour le problème de la congélation d'un liquide saturé dans un réservoir shérique en incluant l'effet du rayonnement à la surface. Les problèmes de frontière mobile de ce type utilisant généralement une analyse de couche limite. Le schéma employé ici évite cette complication et conduit à des estimations successives du temps t_c de solidification complète de la sphère. Une formulation intégrale est adoptée pour établir indépendamment des liens pour t_c. Les liens supérieur et inférieur obtenus sont respectivement l'estimation corrigée d'ordre un et l'esimation de pseudo-état permanent de t_c. Les valeurs numériques des imations successives de t_c montrent une convergence satisfaisante et aussi une cohérence avec les liens établis indépendamment. Les valeurs numériques obtenues pour la position du front mobile sont en accord avec des résultats antérieurs obtenus par une solution complètement numérique et par une solution semi-analytique alternée.

ERSTARREN EINER GESÄTTIGTEN FLÜSSIGKEIT INNERHALB EINER KUGEL

Zusammenfassung—Ein neues halbanalytisches Verfahren wird auf das Problem des Gefrierens einer gesättigten Flüssigkeit in einem kugelförmigen Behälter angewandt, wobei der Strahlungseinfluß der Behälteroberfläche berücksichtigt wird. Im allgemeinen erfordern derartige Probleme mit fortschreitender Grenzfläche eine Grenzschichtanalyse. Das Näherungsverfahren, das hier angewandt wird, vermeidet diese Komplikation und ermöglicht die sukzessive näherungsweise Berechnung der Zeit t_c für die vollständige Verfestigung der Kugel. Zusätzlich wird eine Intergralformulierung herangezogen, um unabhängig die Grenzen für t_c zu beziehnen. Die oberen und unteren Grenzen sind der verbesserte Näherungswert erster Ordnung für t_c beziehungsweise der quasistationäre Näherungswert von t_c. Die numerischen Werte der sukzessiven Näherungsrechnung für t_c zeigen erstens eine befriedigende Konvergenz und stimmen zweitens gut mit den unabhängig ermittelten Grenzen überein. Die Zahlenwerte für die Position der fortschreitenden Erstarrungsfront stehen in Einklang mit früheren Ergebnissen, die einmal aus einer vollständig numerischen Lösung und außerdem aus einer alternativen halbanalytischen Lösung des Problems stammen.

ЗАТВЕРДЕВАНИЕ НАСЫЩЕННОЙ ЖИДКОСТИ ВНУТРИ СФЕРЫ

Аннотация—Используется новая приближенная методика решения задачи затвердевания насыщенной жидкости внутри сферической оболочки с учетом влияния излучения, падающего на ее поверхность. Как правило, при решении задач такого типа с подвижной границей используется приближение пограничного слоя. Применяемая в работе аппроксимационная схема позволяет исключить этот этап и получить последовательные оценки времени t_c полного затвердевания сферы. Кроме того, отдельно методом интегрирования определены пределы значений t_c. Верхний представляет собой обычную скорректированную оценку t_c первого порядка, а нижний – псевдостационарную. Численные значения последовательных оценок t_c свидетельствуют, во-первых, об удовлетворительной сходимости и, во-вторых, о том, что они действительно согласуются с пределами, определяемыми независимым методом. Численные значения, полученные при определении положения движущегося фронта, согласуются с результатами, полученными ранее.